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The terminal velocity of sedimenting particles in a flowing fluid

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Abstract
The influence of an underlying carrier flow on the terminal velocity of sedimenting particles is investigated both analytically and numerically. Our theoretical framework works for a general class of (laminar or turbulent) velocity fields and, by means of an ordinary perturbation expansion at small Stokes number, leads to closed partial differential equations (PDE) whose solutions contain all relevant information on the sedimentation process. The set of PDE is solved by means of direct numerical simulations for a class of 2D cellular flows (static and time dependent) and the resulting phenomenology is analysed and discussed.

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1. Introduction

In many situations of interest, particles suspended in fluids cannot be modelled as simple point tracers. Both drops in gases and bubbles in liquids, and also solid powders in fluids, have a finite size and their density is, generally speaking, different from that of the advecting fluid. The description of their movement must then take into account the effects of inertia: this is why such objects are usually called inertial particles. Understanding the dynamics of these impurities is an open issue from the theoretical, numerical and experimental point of view [1–7], and is very relevant in several domains, ranging from geophysics [8–10] to astrophysics [11, 12], and from industry to biology [13, 14].

Here, our attention will be focused on the sedimentation of inertial particles seeded in a given flow and subject to the action of gravity. Moreover, we will also consider the effect of particle diffusivity. The latter is usually neglected in most studies of inertial particles, assuming that Brownian noise is very small for finite-size particles. However, this is not the case for tiny particles, and, more importantly, in biophysical applications, in which a limited capacity of autonomous movement could e.g. be taken into account in this very simplified way.

Our main aim is to obtain an Eulerian description of the sedimentation process starting from the well-founded Lagrangian viewpoint for particle motion in the limit of small inertia (i.e. when collision events can be neglected). Even if we are not going to deal with multi-particle problems arising at high inertia, such as the formation of caustic patterns and the phenomenon of clustering, in which a Lagrangian dynamical description is necessary to describe collisions, in the concluding section we will be able to sketch a comparison with the 'continuum' description [3], well known in the literature.

Although our main focus is on sedimentation, or in other words on drift (very relevant e.g. within environmental sciences in connection to the problem of aerosol dispersion, and in marine biology in relation to the dynamics of plankton populations), our theory provides the whole detailed statistical information of particle motion. The probability density function (PDF) of having a particle in a given position at a certain time is indeed available from our approach. As a future perspective, we would like to extend our formalism to attack also more general, or higher-order, effects in particle transport, such as eddy diffusivities and the possibility of anomalous transport in the horizontal direction [15, 16].

Our theoretical machinery, which applies for a wide class of velocity fields (either laminar or turbulent) is tested against numerical simulations (both direct numerical simulations and Lagrangian simulations) for a class of 2D cellular flows (static and time dependent). Cellular flows are physically relevant because they are universally used to mimic, e.g., rolls in thermal convection or Langmuir circulation, i.e. the wind-driven helical circulation on the sea surface. This specific example offers the possibility of analysing and discussing how the sedimentation process turns out to be extremely sensitive to the flow details. To be more specific, for the class of cellular flows, both an increase and a reduction of the rate of sedimentation with respect to the still-fluid case can occur, depending on the flow parameters such as, e.g., flow geometry and pulsating frequency, as well as on the usual fluid-dynamics adimensional numbers which measure the density ratio and the effects of inertia (Stokes), gravity (Froude) and diffusivity (Péclet). Our findings lead to the following conclusion of interest in the realm of applications: any attempt to model the effect of flows on particle sedimentation cannot avoid taking into account the full details of the flow field.

This paper is organized as follows. In section 2 the basic equations governing the time evolution of inertial particles under gravity and in a prescribed flow are given, together with the associated Fokker–Planck equation for the particle PDF. In section 3, the problem of finding the terminal velocity of the sedimenting particles is formulated in a Hermitian form and tackled via a second-quantization formalism. The resulting set of auxiliary equations to obtain the terminal velocity is presented in the limit of small Stokes numbers. In section 4, the auxiliary equations are solved via direct numerical simulations in two dimensions and the obtained results are discussed. Conclusions and perspectives follow in section 5. The appendix is devoted to some mathematical and computational details.

2. General equations

Let us consider the motion of a single, small, rigid, spherical particle of radius b immersed in an incompressible d-dimensional flow $\boldsymbol{u}(\boldsymbol{x}, t)$ (with $d \geqslant 2$). Even if some of our results are more general, the flow field will be assumed either steady or periodic in time (with period T), and periodic in space (i.e. cellular, with period of order L) [17, 18]. We shall focus our attention on the so-called Stokes regime, in which the surrounding flow is differentiable on scales of the order of b; we shall also neglect the feedback of the particle on the flow. The motion is thus influenced by gravity, buoyancy and drag [19], to which Brownian noise should be added in order to take into account the thermal fluctuations of the fluid, or alternatively (if under investigation are micro-organisms, considered as inertial particles but with a limited self-capacity of moving) to mimic disorganized movement of small biological entities e.g. in the ocean. Moreover, we neglect the Basset, Faxen, Oseen and Saffman corrections [19] and other possible effects due to rotationality, high relative velocity or non-sphericity [20].

Lastly, the well-known added-mass effect will be taken into account in a simplified way. To understand this point, one should remember that the evolution equation for the particle trajectory $\boldsymbol{X}(t)$ is simply Newton's law,

$$\ddot{\boldsymbol{X}}(t) = \cdots + c_{\mathrm{d}} \frac{\mathrm{d}\boldsymbol{u}(\boldsymbol{X}(t), t)}{\mathrm{d}t} + c_{\mathrm{D}} \frac{\mathrm{D}\boldsymbol{u}(\boldsymbol{X}(t), t)}{\mathrm{D}t}, \tag{1}$$

in which the force exerted by the fluid on the particle must include the fact that some fluid is moved by the particle itself during its motion. This is why, on the right-hand side of (1), there appears the time derivative of $\boldsymbol{u}$. The latter can be interpreted as a derivative following either the particle trajectory, $\mathrm{d}/\mathrm{d}t \equiv \partial_t + \dot{\boldsymbol{X}} \cdot \boldsymbol{\partial}$, or the fluid path, $\mathrm{D}/\mathrm{D}t \equiv \partial_t + \boldsymbol{u} \cdot \boldsymbol{\partial}$. Several works dealt with this aspect [19, 21–23] and explained the historical evolution and the different roles played by the coefficients c_{d} and c_{D}. In the present work, for the sake of simplicity, we assume $c_{\mathrm{D}} = 0$ and, consequently, the whole effect is included into c_{d}.

In this framework, it is customary to recast (1) from its original second-order formulation into a dynamical system consisting of two coupled differential equations, by introducing the covelocity $\boldsymbol{V} \equiv \dot{\boldsymbol{X}} - \beta \boldsymbol{u}(\boldsymbol{X}(t), t)$. Here, $\beta \equiv 3\rho_{\mathrm{f}}/(\rho_{\mathrm{f}} + 2\rho_{\mathrm{p}})$ is an adimensional coefficient, where ρ_{p} and ρ_{f} are the densities of the particle and the fluid, respectively. According to the ratio between the two densities, β ranges from 0 ($\rho_{\mathrm{f}} \ll \rho_{\mathrm{p}}$: heavy particles, like drops in gases) to 3 ($\rho_{\mathrm{f}} \gg \rho_{\mathrm{p}}$: light particles, like bubbles in liquids), and becomes 1 when the two densities are equal (and buoyancy effects absent). Note that the covelocity differs from the slip (relative) velocity ($\equiv \dot{\boldsymbol{X}}(t) - \boldsymbol{u}(\boldsymbol{X}(t), t)$) by a term $(1 - \beta)\boldsymbol{u}(\boldsymbol{X}(t), t)$, which vanishes only for neutral particles. The concept of slip velocity is probably more familiar in connection with the study of relative acceleration. In our case, the use of covelocity is due to the fact that it strongly simplifies the equations of motion, properly keeping into account the added-mass effect without any need to introduce time derivatives of the external flow. However, $\boldsymbol{u}(\boldsymbol{x}, t)$ being completely known, from the knowledge of $\boldsymbol{V}$ one can immediately find the real velocity $\dot{\boldsymbol{X}}$.

The Stokes time is defined as $\tau \equiv b^2/3\nu\beta$, with ν being the kinematic viscosity. It represents the typical response time of the particle to changes in the external flow. Note that a tracer particle, usually referred to as a passive scalar, is characterized not only by $\beta = 1$ (which, alone, is not sufficient to rule out possible inertial effects due to the finite size of the particle) but also by $\tau = 0$.

Denoting by $\boldsymbol{g}$, κ and $\boldsymbol{\eta}(t)$ the gravity acceleration, the particle diffusivity and the standard white noise (stationary Gaussian process defined by $\langle \boldsymbol{\eta} \rangle = 0$ and $\langle \eta_\mu(t)\eta_\nu(0) \rangle = \delta_{\mu\nu}\delta(t)$), respectively, we have

$$\begin{cases} \dot{\boldsymbol{X}} = \boldsymbol{V} + \beta \boldsymbol{u}(\boldsymbol{X}(t), t) \\ \dot{\boldsymbol{V}} = \dfrac{(1 - \beta)\boldsymbol{u}(\boldsymbol{X}(t), t) - \boldsymbol{V}}{\tau} + (1 - \beta)\boldsymbol{g} + \dfrac{\sqrt{2\kappa}}{\tau}\boldsymbol{\eta}(t). \end{cases} \tag{2}$$

Note that, for dimensional consistency, the random term requires the presence of a time scale at the denominator. The latter has been chosen as τ itself for the sake of simplicity and convenience. In this way, in the passive-scalar limit $\tau \to 0$, equations (2) reduce to the well-known expression $\dot{\boldsymbol{X}}(t) = \boldsymbol{u}(\boldsymbol{X}(t), t) + \sqrt{2\kappa}\boldsymbol{\eta}(t)$.

The study can be carried on in the corresponding phase space $(\boldsymbol{x}, \boldsymbol{v}, t)$. The full phase-space approach is needed for inertial particles because the dynamics remains second-order, and there is no collapse onto a first-order problem unlike for the passive scalar. Let us consider the probability density that, at time t, the particle is at location $\boldsymbol{x}$ with a covelocity $\boldsymbol{v}$, and denote it by $p(\boldsymbol{x}, \boldsymbol{v}, t)$. This satisfies the Fokker–Planck equation associated with the stochastic differential equation (2):

$$\mathcal{L}p \equiv \left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial x_\mu}[v_\mu + \beta u_\mu(\boldsymbol{x}, t)] + \frac{\partial}{\partial v_\mu}\left[(1-\beta)g_\mu + \frac{(1-\beta)u_\mu(\boldsymbol{x}, t) - v_\mu}{\tau}\right] - \frac{\kappa}{\tau^2}\frac{\partial^2}{\partial v_\mu \partial v_\mu}\right\} p = 0. \tag{3}$$

The linear operator $\mathcal{L}$ inside the curly brackets of (3) carries a factor κ/τ^2 in front of the Laplacian with respect to the velocity, which is the highest-derivative term. Therefore, we may expect some singularity in the limits of vanishing diffusivity or large Stokes time.

The terminal settling velocity of the inertial particle [24, 25] is defined as the average particle velocity

$$\boldsymbol{w} \equiv \frac{1}{T}\int \mathrm{d}t \int \mathrm{d}\boldsymbol{x} \int \mathrm{d}\boldsymbol{v}[\boldsymbol{v} + \beta \boldsymbol{u}(\boldsymbol{x}, t)]p(\boldsymbol{x}, \boldsymbol{v}, t), \tag{4}$$

where the temporal integration is clearly omitted in the presence of steady flows[1].

In general, such *renormalized* terminal velocity can differ from the *bare* terminal velocity, i.e. the particle settling velocity in a still fluid, whose value can easily be found to be

$$\boldsymbol{w}^* = (1-\beta)\boldsymbol{g}\tau. \tag{5}$$

The difference turns out to be

$$\Delta\boldsymbol{w} \equiv \boldsymbol{w} - \boldsymbol{w}^* = T^{-1}\int \mathrm{d}t \int \mathrm{d}\boldsymbol{x} \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{u}(\boldsymbol{x}, t)p(\boldsymbol{x}, \boldsymbol{v}, t). \tag{6}$$

In what follows, we shall only take into account flows with zero average (i.e., $\int \mathrm{d}t \int \mathrm{d}\boldsymbol{x}\, \boldsymbol{u}(\boldsymbol{x}, t) = 0$) and possessing odd parity with respect to reflections in the vertical direction. Such flows are relevant to analyse the vertical component Δw of the terminal velocity discrepancy because no mean contribution is present, and every (eventual) nonzero result, originated from the component of p antisymmetric in the vertical coordinate, is to be interpreted as due to preferential concentration in areas of rising or falling fluid.

Our plan is thus to solve (3), $\mathcal{L}p = 0$, for the phase-space density and to plug the result in expression (6) to find Δw.

3. Analytical investigation at small Stokes number

We shall first focus on situations in which the Stokes time is much smaller than the flow typical time scale, so that the particle adapts its own velocity to that of the underlying fluid very rapidly. More precisely, denoting by L and U the typical length scale and the typical velocity of the fluid respectively, in this section we shall only deal with small Stokes number, $\mathrm{St} \equiv \tau/(L/U)$. As we also take into account gravity and diffusivity (besides inertia), a thorough adimensionalization of the problem also requires the introduction of the Froude and Péclet numbers, defined as $\mathrm{Fr} \equiv U/\sqrt{gL}$ and $\mathrm{Pe} \equiv LU/\kappa$, respectively.

[1] It is worth noting that expression (4) can also be deduced by means of a multiscale expansion [26–29], imposing a diffusive rescaling for space and time and considering covelocity exclusively as a fast variable. At the first order in the expansion, the average settling velocity (4) emerges naturally as the quantity which must vanish when imposing the solvability condition, necessary to proceed to higher orders. In other words, to obtain an effective-diffusivity behaviour at the second order of the expansion, one should perform a Galilean transformation into a frame of reference such as to eliminate any drift. This will be the subject of our future investigation.

Note that our choice of the denominator in St (the advective time scale L/U) is 'universal' for steady laminar flows, while for time-dependent flows other possibilities could also hold, e.g. the period T or the inverse of the Lyapunov exponent. Our choice is the simplest one because L and U are the quantities which also appear in Fr and Pe, while the other options may also bring into play other adimensional quantities such as the Kubo or Strouhal numbers. Moreover, according to our adimensionalization, it is possible to study the effect of an oscillation frequency at fixed St leaving the particle response time τ unchanged (note, on the other hand, that this is not the case when investigating the role of β: see remark (ii) near the end of this section).

As a first result, we see immediately that the vertical component of the bare terminal velocity (5), assumed as positive if pointing downwards and written in units of U, is rewritten as

$$w^* = (1-\beta)\mathrm{St}\,\mathrm{Fr}^{-2}. \tag{7}$$

Let us adimensionalize $\boldsymbol{x}$ with L, t with L/U and $\boldsymbol{u}$ with U, and let us denote the new variables with the same letters as before. Equation (3) can now be written in an adimensional form after introducing an appropriate variable for the rescaled covelocity: the correct one turns out to be $\boldsymbol{y} \equiv \sqrt{\tau/2\kappa}\,\boldsymbol{v}$. With this notation, the operator acting upon the density $p(\boldsymbol{x},\boldsymbol{y},t)$ (appropriately normalized in the new variables) becomes:

$$\begin{aligned}\mathcal{L} &= -\mathrm{St}^{-1}\left[\frac{\partial}{\partial y_\mu}y_\mu + \frac{1}{2}\frac{\partial^2}{\partial y_\mu \partial y_\mu}\right] + \mathrm{St}^{-1/2}\left[\sqrt{\frac{2}{\mathrm{Pe}}}\,y_\mu\frac{\partial}{\partial x_\mu} + \sqrt{\frac{\mathrm{Pe}}{2}}(1-\beta)u_\mu(\boldsymbol{x},t)\frac{\partial}{\partial y_\mu}\right] \\ &\quad + \mathrm{St}^0\left[\frac{\partial}{\partial t} + \beta u_\mu(\boldsymbol{x},t)\frac{\partial}{\partial x_\mu}\right] + \mathrm{St}^{1/2}\left[\sqrt{\frac{\mathrm{Pe}}{2}}\frac{1-\beta}{\mathrm{Fr}^2}G_\mu\frac{\partial}{\partial y_\mu}\right] \\ &\equiv -\mathrm{St}^{-1}\mathcal{A}_0 + \mathrm{St}^{-1/2}\mathcal{A}_1 + \mathrm{St}^0\mathcal{A}_2 + \mathrm{St}^{1/2}\mathcal{A}_3,\end{aligned} \tag{8}$$

where $\boldsymbol{G} \equiv \boldsymbol{g}/g$ is the unit vector pointing downwards, and the expressions of the operators $\mathcal{A}_i$ ($i = 0, \ldots, 3$) are easily identified from (8). It is worth noting that $\mathcal{A}_0$ is an Ornstein–Uhlenbeck operator acting on covelocity coordinates only, which leads to a Hermitian reformulation of the problem in terms of a second-quantization formalism for the covelocity variable. Technical details on the analytical computation are left to the appendix: here we only recall the essential steps. By introducing a power-series expansion in St for the particle PDF, we obtain a chain of advection–diffusion equations (see (10)) for the auxiliary quantities $\psi_n^{\emptyset}(\boldsymbol{x},t)$ ($n \in \mathbb{N}$) defined in the appendix, in which lower-order quantities act as source terms for higher-order ones. Such equations can be solved sequentially, at least numerically, once the external flow is given. As a result, expression (6) for the variation of terminal velocity (in units of U) is rewritten as

$$\Delta\boldsymbol{w} = \left(\frac{T}{L/U}\right)^{-1}\sum_{m=0}^{\infty}\mathrm{St}^{2+m}\int \mathrm{d}t\int \mathrm{d}\boldsymbol{x}\,\boldsymbol{u}(\boldsymbol{x},t)\psi_{4+2m}^{\emptyset}(\boldsymbol{x},t), \tag{9}$$

where the integral on covelocity has already been performed (see (A.8)). Note that only the vertically-antisymmetric components of $\psi_{4+2m}^{\emptyset}(\boldsymbol{x},t)$, hereafter denoted by the superscript $^{(\mathrm{a})}$ (in order to distinguish them from the vertically-symmetric counterparts $^{(\mathrm{s})}$), contribute to the spatial integral in (9) for the vertical component Δw. Therefore, in order to study the leading contribution in Δw, quadratic in St (the cumbersome equations needed to compute $\mathrm{O}(\mathrm{St}^3)$ are reported in the appendix), it is sufficient to consider the system

$$\begin{cases} \mathcal{M}\psi_4^{\emptyset(\mathrm{a})} = -(1-\beta)\mathrm{Fr}^{-2}G_\mu \dfrac{\partial}{\partial x_\mu}\psi_2^\emptyset \\ \mathcal{M}\psi_2^\emptyset = (1-\beta)\dfrac{\partial u_\mu}{\partial x_\nu}\dfrac{\partial u_\nu}{\partial x_\mu}\psi_0^\emptyset \Longrightarrow \psi_2^\emptyset = \psi_2^{\emptyset(\mathrm{s})} \\ \mathcal{M}\psi_0^\emptyset = 0 \Longrightarrow \psi_0^\emptyset = \mathrm{const} = \psi_0^{\emptyset(\mathrm{s})}, \end{cases} \tag{10}$$

where we imposed uniform initial conditions (such that $\psi_0^\emptyset$ is a normalization constant) and denoted the adimensionalized advection–diffusion operator with

$$\mathcal{M} \equiv \frac{\partial}{\partial t} + u_\mu(\boldsymbol{x}, t)\frac{\partial}{\partial x_\mu} - \mathrm{Pe}^{-1}\frac{\partial^2}{\partial x_\mu \partial x_\mu}. \tag{11}$$

From (10), it is easy to understand that both $\psi_0^\emptyset$ and $\psi_2^\emptyset$ do not depend on Fr, and the former is clearly also independent of β. This means that $\psi_2^\emptyset$ behaves like $(1-\beta)$, therefore $\psi_4^{\emptyset(\mathrm{a})}$ is proportional to both $(1-\beta)^2$ and Fr^{-2}.

Let us now focus on this leading contribution at order St^2 in Δw, corresponding to the term $m = 0$ in (9). From the analytical point of view, such considerations imply that we can introduce a non-universal (in the sense that it depends on Pe and possibly on other properties of the surrounding flow, e.g. its time frequency or space geometry, but not on St, Fr or β) function

$$\Delta W \equiv \frac{L}{UT}\int \mathrm{d}t \int \mathrm{d}\boldsymbol{x}\, G_\mu u_\mu(\boldsymbol{x}, t)\frac{\psi_4^{\emptyset(\mathrm{a})}(\boldsymbol{x}, t)}{\mathrm{Fr}^{-2}(1-\beta)^2}, \tag{12}$$

such that we can write

$$\Delta w = [(1-\beta)\mathrm{St}/\mathrm{Fr}]^2 \Delta W + \mathrm{O}(\mathrm{St}^3). \tag{13}$$

Equation (13) summarizes the following two main analytical results for the variation of the actual terminal velocity in the limit of small inertia:

(i) It is influenced by gravity in the same way as the bare terminal velocity. Indeed, both w^* in (7), and the coefficient in square brackets beside ΔW in (13), are proportional to Fr^{-2}, i.e. to $\boldsymbol{g}$. This result is not trivial, because e.g. it does not hold, in general, at higher orders in St, nor, consequently, for the full renormalized terminal velocity.

(ii) It has the opposite parity, in $(1-\beta)$, with respect to the bare terminal velocity. Indeed, w^* scales with $(1-\beta)$ while the above-mentioned coefficient with its square. In other words, if a specific flow under specific conditions is found to increase the falling of heavy particles, then the same flow under the same conditions (e.g. the same St, which may imply different L/U because τ depends on β) must simultaneously decrease the rising of light particles, and vice versa.

The dependence on molecular diffusivity is more difficult to obtain analytically. In particular, as already pointed out, the large-Pe limit is singular, and nothing precise can be said about intermediate Pe. In contrast, for small Pe, it can be shown rigorously that the behaviour of ΔW is $\mathrm{O}(\mathrm{Pe}^2)$ or higher. As a particular case, we will show in section 4 that, for the 2D stationary square-cell flow defined in (14), very well-defined asymptotics are found numerically for both small and large Pe. In any case, one should keep in mind that ours is a series expansion in powers of St, whose coefficients may become very large when varying the other parameters. In order for this to work, St should be assumed small enough so as to make the following terms negligible with respect to the previous ones.

4. Numerical investigation: the cellular flow

From the analytical point of view, the results reported in section 3 are the only available, and all refer to the $O(\mathrm{St}^2)$ correction. Proceeding at higher orders is not an easy task: e.g., the equation in (10) for $\psi_6^{\emptyset(a)}$ is much more complicated (see (A.7)), and involves as a source term also the vertically-symmetric component $\psi_4^{\emptyset(s)}$, whose equation is in turn by far more cumbersome than that for the corresponding antisymmetric part. Clearly, this problem also reflects on the numerical approach: solving such equations by direct numerical simulation (DNS) is beyond our possibilities. The DNS approach can however be applied to $\psi_2^{\emptyset}$ and $\psi_4^{\emptyset(a)}$ if the surrounding flow is simple enough.

An accurate study has been performed for the 2D cellular flow:

$$\begin{cases} u_1 = \sin(kx_1)\cos[x_2 + B\sin(\omega t)] \\ u_2 = -k\cos(kx_1)\sin[x_2 + B\sin(\omega t)], \end{cases} \tag{14}$$

where the parameters k, B and ω represent respectively the cell aspect ratio (vertical extension over the horizontal one) and the amplitude and pulsation of the oscillations in the vertical direction x_2, assumed as positive downwards. In other words, in usual dimensional variables, the cell has a horizontal extension of $2\pi L/k$ and a vertical one of $2\pi L$, and oscillates vertically with spatial excursion $\pm BL$ and period $T = 2\pi L/U\omega$.

Such a synthetic flow, introduced as an instance mimicking Langmuir circulation or Rayleigh–Bénard convection [30], has then been used for both numerical and analytical studies in several domains, ranging from front propagation to polymer transport, and from anomalous diffusion to chaotic maps [15, 18, 23, 31–35], because of its simplicity and its ability to capture different physical behaviours.

The figures in this section all refer to the flow described by (14) for heavy particles ($\beta = 0$) with $B = 2$, and (except for a couple of forthcoming aside remarks) in square cells ($k = 1$) at the specific value $\mathrm{Fr} = 1$.

Our numerical procedure mainly consists in comparing the outcomes of two different ways of computing the vertical component of the terminal velocity. On one hand, we performed a Lagrangian simulation (LS) of the particle dynamics (2), as a function of St, at the fixed value $\mathrm{Pe} = 5$ and for two different values of the frequency ($\omega = 0$ and $\omega = 0.6$). In this scheme (see, e.g., [36], which however applies to a somehow different situation) the single particle is followed and the external flow is 'sampled' only along the particle trajectory; the Milstein method is used for the integration. As long as equations (2) hold, which is especially true for our case $\beta = 0$, the outcome of our LS can thus be thought of as the real particle evolution[2]. On the other hand, we also performed a high-resolution (1024^2) DNS of the field equations (10), and then we computed the sedimentation velocity using (9) or equivalently, at this order in St, (12). The numerical integration consisted of a standard 2/3-dealiased pseudospectral method with time evolution implemented by a standard second-order Runge–Kutta scheme. The results of the DNS are then used both to verify the validity of our 'field' approach truncated at the quadratic approximation in St (by comparing the outcomes of DNS and LS at the same values of the other parameters, as in figures 1 and 3), and to obtain a numerical description of the role played, e.g., by Pe (figure 2) and ω (figure 4) in the expression of ΔW in (13).

[2] In order to test the importance of conclusion (i) of section 3, we verified through LS that the full renormalized terminal velocity does not rescale simply with Fr^{-2}, differently from its bare value and its leading correction at $O(\mathrm{St}^2)$.

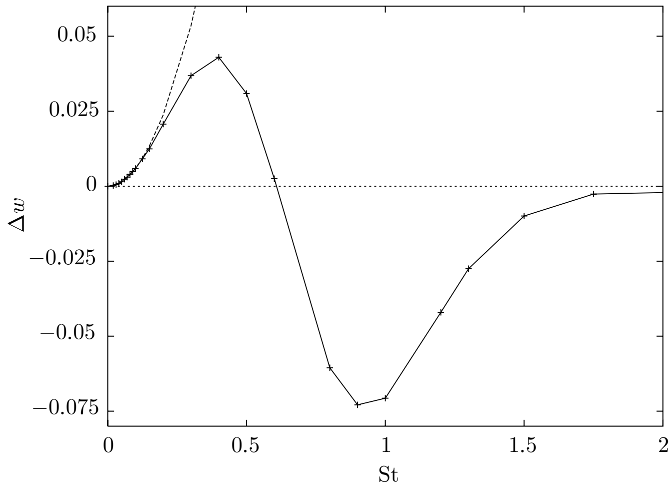

Figure 1. Comparison between LS (solid line) of (2) and the parabola (dashed line) resulting from (12) using the DNS solution of (10) for the variation of the falling velocity as a function of the Stokes number at $\omega = 0$.

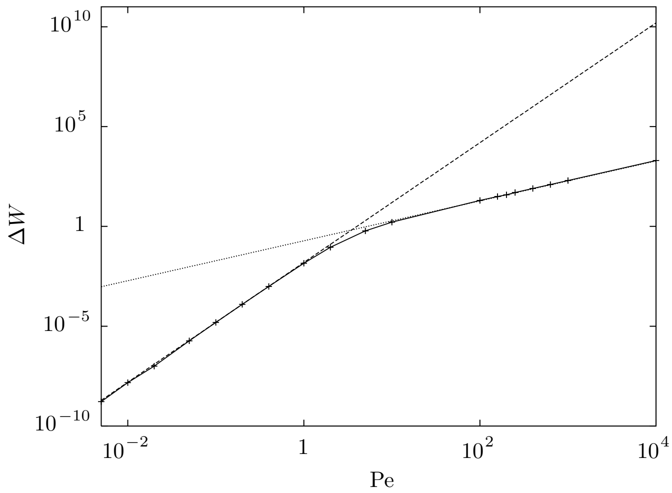

Figure 2. Dependence on the Péclet number for the variation of the falling velocity (solid line) at $O(St^2)$ from DNS. The dashed line represents the cubic asymptote found analytically, while the dotted line is the linear asymptote with a multiplicative coefficient derived from a fit.

First, we test our analytical result on the $O(St^2)$ correction in the steady case ($\omega = 0$). To do this, we performed a LS of (2) at $Pe = 5$,[3] and solved the system (10) for $\psi_4^{\emptyset(a)}$ (and for $\psi_2^{\emptyset}$ in parallel) by means of DNS. In figure 1, for small (and intermediate) values of St, we report such LS and the parabola $St^2 \Delta W$, where ΔW is computed through (12) with the $\psi_4^{\emptyset(a)}$ found from DNS: the agreement is very good until about $St = 0.2$. Moving at larger St, it is evident how the increase in the falling velocity reaches a maximum value at $St \simeq 0.4$, then there

[3] Also the value $Pe = 1000$ was tested with qualitatively similar results.

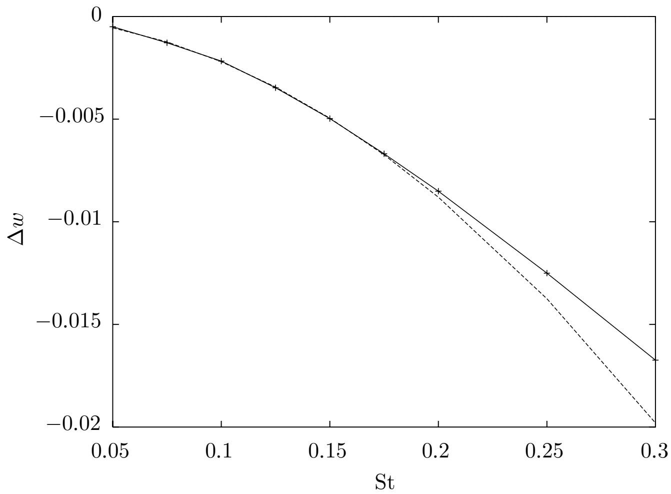

Figure 3. Same as in figure 1 but with $\omega = 0.6$. Note the decrease in the falling velocity opposed to the static-carrier-flow case.

is a crossover and a maximum reduction in settling takes place at St $\simeq 1$. The asymptotic vanishing at large St is a general feature in the ballistic limit, in which the particle reacts so slowly to the variations of the surrounding flow that the latter has basically no influence on settling. Such a limit is singular, as already pointed out, because a particle with infinite inertia in a finite-correlated flow is equivalent to a particle with finite inertia in a δ-correlated flow [37, 38], thus a technique like uniform-coloured-noise approximation should be used to deal with this situation properly. In contrast, we think that our perturbative approach at small St should be able to capture at least the maximum in the plot, taking into account the following orders like $\psi_6^{\emptyset(\mathrm{a})}$ or higher. Of course, this behaviour could not be caught correctly by our previous quadratic approximation, which is not able to describe inflection points and subsequent changes in convexity.

Let us now turn to the dependence of ΔW on Pe. As already mentioned at the end of section 3, two well-defined asymptotics can be found, as shown in figure 2. In particular, at small Pe, it can be shown by means of perturbative analysis that ΔW approaches $f(k)\mathrm{Pe}^3$, where $f(k) = (k^4+3k^2-2)/32(k^2+1)^2$ (i.e., $f(k) \geqslant 0 \Leftrightarrow k \geqslant (\sqrt{17}-3)/2 \simeq 0.56$).[4] Note that this result is more stringent than the general behaviour $\mathrm{O}(\mathrm{Pe}^2)$ (or higher) reported at the end of section 3, and only applies to this particular instance of flow. In contrast, the linear asymptote 0.19 Pe at large Pe comes from a fitting procedure, because this limit is singular. Note, however, that this does not necessarily imply that the terminal velocity diverges at infinite Pe: what we are discussing in this paragraph is only the behaviour of the $\mathrm{O}(\mathrm{St}^2)$ term, and clearly there is the possibility of a regularizing resummation if one also considers higher-order terms.

In figure 3 we show that for a nonzero pulsating frequency the effect of the carrier flow may be the opposite. Indeed, at $\omega = 0.6$, the variation of the terminal velocity from LS is now negative at small inertia, and our perturbative field approach (the dashed parabola with

[4] As an example of the role played by the cell aspect ratio k, we found from DNS that ΔW can have the opposite sign when under consideration are cells not square but rather rectangular (extended horizontally) with $k = 1/2$.

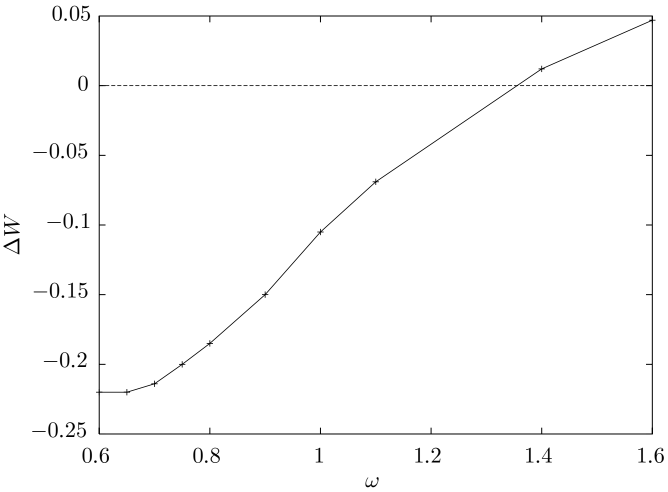

Figure 4. Dependence on the pulsation ω for the variation of the falling velocity (solid line) at $O(\mathrm{St}^2)$, with $\mathrm{Pe} = 5$.

coefficient ΔW computed through DNS) again provides a good approximation at least until $\mathrm{St} = 0.2$.

This is confirmed by the fact that, analysing ΔW as a function of ω through our DNS (see figure 4), we find a reduction in settling in a window of values of ω. For larger frequencies ($\omega > 1.4$) an increase in the sedimenting velocity occurs again. Moreover, in the limit $\omega \to \infty$, the effect of the flow on the inertial particle is generally expected to vanish, because the latter does not have enough time to adapt its own velocity to the one it feels around itself, which varies very rapidly.

This clearly shows how sensitive to the flow details the terminal velocity may be.

5. Conclusions

We have investigated how the terminal velocity of sedimenting particles is influenced by a background flow. In the limit of small St, we have reduced this problem to the solution of two coupled forced advection–diffusion equations. Such equations hold for generic velocities, laminar or turbulent, possessing odd parity with respect to reflections in the vertical direction. From our analysis the complete knowledge of the particle motion is available: the PDF of having a particle in a given position at a given time is directly accessible once the differential equations are solved. The determination of the terminal velocity is a particular product of our analysis. Apart from the limiting case of small Péclet numbers (which may be relevant e.g. in the already-mentioned case of self-movement of micro-organisms mimicked by Brownian noise), one has to resort to numerical simulations to find the PDF and the consequent terminal velocity. This task is easily accomplished for two-dimensional laminar flows, which clearly lead to the following conclusions. The value of St alone is not sufficient to argue if the sedimentation is faster or slower with respect to what happens in still fluid. In particular, for square, static cellular flows, at both intermediate and high Pe, we found an increase of the falling velocity at relatively small St, and a reduction starting from St larger than some critical value. A similar transition is also found for a fixed St and assuming the cell to be oscillating in the vertical direction with frequency ω. We found the remarkable result that, appropriately tuning ω, a decrease in settling can also be obtained at small St, accompanied by an increase now occurring at sufficiently large St. We also studied the effect of the flow spatial geometry on the terminal velocity. In the limit of small Pe, the expression for the terminal velocity can be extracted analytically from our equations and its explicit dependence on the cell aspect ratio isolated. Changing the cell aspect ratio appropriately can cause a reduction in falling at small St.

As a last point, it is now possible to make a comparison between our method and the well-known 'continuum' approximation [3] at small St. The latter neglects diffusivity in (2) and, by recursive substitution, obtains a field description of the covelocity:

$$\begin{aligned}\boldsymbol{v}(\boldsymbol{x},t) &= (1-\beta)\boldsymbol{u}(\boldsymbol{x},t) + \mathrm{St}(1-\beta)[\mathrm{Fr}^{-2}\boldsymbol{G} - (\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\partial}\boldsymbol{u})] + \mathrm{O}(\mathrm{St}^2)\\ &= \boldsymbol{v}_{(0)} + \mathrm{St}\boldsymbol{v}_{(1)} + \sum_{l=2}^{\infty}\mathrm{St}^l\boldsymbol{v}_{(l)},\end{aligned} \tag{15}$$

where $\boldsymbol{v}_{(l)} = (1-\beta)(-\partial_t - \boldsymbol{u}\cdot\boldsymbol{\partial})^l\boldsymbol{u}$ for $l \geqslant 2$. The relevant equation for the physical-space density $\rho(\boldsymbol{x},t)$ is simply the continuity equation, with velocity given by $\boldsymbol{v}(\boldsymbol{x},t) + \beta\boldsymbol{u}(\boldsymbol{x},t)$:

$$\partial_t\rho + \boldsymbol{\partial}\cdot[(\boldsymbol{v}+\beta\boldsymbol{u})\rho] = 0 \tag{16}$$

(here we neglect a possible diffusive term for the sake of simplicity in the comparison). By plugging (15) in (16), one finds a system of equations for the coefficients $\rho_{(l)}(\boldsymbol{x},t)$ of the power-series expansion $\rho = \sum_{l=0}^{\infty}\mathrm{St}^l\rho_{(l)}$. A clear parallelism holds between each $\rho_{(l)}(\boldsymbol{x},t)$ and the corresponding quantity of our formalism, $\psi_{2l}^{\emptyset}(\boldsymbol{x},t)$: indeed, to find the variation of the vertical terminal velocity at order l in St, one has to perform a spatial–temporal integration of the vertical fluid velocity times (the vertically-antisymmetric component of) either of these two quantities. The comparison can be carried out by comparing the system of equations for $\rho_{(l)}$ with that for $\psi_{2l}^{\emptyset}$ after setting diffusivity to zero in the latter. The result is that $\rho_{(0)}$, $\rho_{(1)}$ and $\rho_{(2)}^{(a)}$ obey the same equations (10) as $\psi_0^{\emptyset}$, $\psi_2^{\emptyset}$ and $\psi_4^{\emptyset(a)}$, respectively, without the diffusive term. However, it turns out that at higher orders in St the two approaches differ: e.g., both $\rho_{(2)}^{(s)}$ and $\rho_{(3)}^{(a)}$ satisfy equations completely different from the ones (A.7) for the corresponding quantities $\psi_4^{\emptyset(s)}$ and $\psi_6^{\emptyset(a)}$ reported in the appendix. Therefore, even if the same conclusions drawn at $\mathrm{O}(\mathrm{St}^2)$ on the dependence of ΔW could also be found with the continuum approach, we can assert that our rigorous approach proves fruitful because it represents the only correct way to attack higher orders ($\geqslant$3) in St.

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Appendix. Mathematical and computational details

In this section we provide some details of the calculation which leads to (10).

A Hermitian reformulation of the problem is convenient. Let us define $p_\dagger(\boldsymbol{y}) \equiv \exp(-y^2)$, the Gaussian kernel of $\mathcal{A}_0$: $\mathcal{A}_0 p_\dagger = 0$. After decomposing the particle PDF as

$$p(\boldsymbol{x},\boldsymbol{y},t) = p_\dagger^{1/2}(\boldsymbol{y})\psi(\boldsymbol{x},\boldsymbol{y},t), \tag{A.1}$$

our aim is to find the operators $\mathcal{B}_i$ such that $\mathcal{A}_i p = -p_\dagger^{1/2}\mathcal{B}_i\psi$ $(i = 0,\ldots,3)$.

This can be achieved by means of a second-quantization algorithm. One can indeed introduce the operators of creation and annihilation

$$a_\mu^\pm = y_\mu \mp \frac{\partial}{\partial y_\mu},$$

in terms of which

$$y_\mu = \frac{1}{2}\big(a_\mu^+ + a_\mu^-\big), \qquad \frac{\partial}{\partial y_\mu} = \frac{1}{2}\big(a_\mu^- - a_\mu^+\big)$$

and the vacuum state turns out to be $a_\mu^-|0\rangle = 0 \Leftrightarrow |0\rangle = p_\dagger^{1/2}$.

Defining

$$\alpha_\mu = \sqrt{\frac{\mathrm{Pe}}{2}}(1-\beta)u_\mu(\boldsymbol{x},t) - \frac{1}{\sqrt{2\mathrm{Pe}}}\frac{\partial}{\partial x_\mu},$$

$$\gamma_\mu = -\frac{1}{\sqrt{2\mathrm{Pe}}}\frac{\partial}{\partial x_\mu}, \qquad \zeta_\mu = \sqrt{\frac{\mathrm{Pe}}{2}}\frac{1-\beta}{\mathrm{Fr}^2}G_\mu,$$

the operators we are looking for read:

$$\mathcal{B}_0 = \frac{1}{2}\left(y^2 - \frac{\partial^2}{\partial y_\mu \partial y_\mu} - d\right) = \frac{1}{2}a_\mu^+ a_\mu^-, \qquad \mathcal{B}_1 = \alpha_\mu a_\mu^+ + \gamma_\mu a_\mu^-,$$

$$\mathcal{B}_2 = -\frac{\partial}{\partial t} - \beta u_\mu(\boldsymbol{x},t)\frac{\partial}{\partial x_\mu}, \qquad \mathcal{B}_3 = \zeta_\mu a_\mu^+.$$

Note that $\mathcal{B}_0$ can be interpreted as the occupation number, and the following commutation relations hold:

$$\big[\mathcal{B}_0, a_\mu^\pm\big] = \pm a_\mu^\pm, \qquad \big[a_\mu^-, a_\nu^+\big] = 2\delta_{\mu\nu}, \qquad \big[a_\mu^\pm, a_\nu^\pm\big] = 0.$$

On the other hand, $\mathcal{B}_2$ does not carry any creation or annihilation operator ($\mathcal{A}_2$ is independent of $\boldsymbol{y}$) and vanishes when under investigation are very heavy particles in a stationary state.

The following step consists in the development

$$\psi(\boldsymbol{x},\boldsymbol{y},t) = \sum_{n=0}^{\infty} \mathrm{St}^{n/2}\psi_n(\boldsymbol{x},\boldsymbol{y},t) \tag{A.2}$$

and in the study of the set of equations arising at each (integer and half-i.) order in St:

$$\mathcal{B}_0\psi_n = \begin{cases} 0 & \text{for } n=0 \\ \mathcal{B}_1\psi_{n-1} & \text{for } n=1 \\ \mathcal{B}_1\psi_{n-1} + \mathcal{B}_2\psi_{n-2} & \text{for } n=2 \\ \mathcal{B}_1\psi_{n-1} + \mathcal{B}_2\psi_{n-2} + \mathcal{B}_3\psi_{n-3} & \text{for } n\geqslant 3. \end{cases} \tag{A.3}$$

Such relations can be solved recursively by exploiting the simple inversion formula for a generic function Ψ:

$$\mathcal{B}_0\Psi = a_{\mu_k}^+ \cdots a_{\mu_1}^+|0\rangle \quad\Longrightarrow\quad \Psi = \frac{1}{k}a_{\mu_k}^+\cdots a_{\mu_1}^+|0\rangle.$$

Therefore, one obtains

$$\psi_n(\boldsymbol{x},\boldsymbol{y},t) = \psi_n^{\emptyset}(\boldsymbol{x},t)|0\rangle + \psi_n^{\mu_1}(\boldsymbol{x},t)a_{\mu_1}^+|0\rangle + \cdots + \frac{1}{n!}\psi_n^{\mu_1\cdots\mu_n}(\boldsymbol{x},t)a_{\mu_n}^+\cdots a_{\mu_1}^+|0\rangle, \tag{A.4}$$

where

$$\psi_n^{\mu_1\cdots\mu_k}(\boldsymbol{x},t) = \begin{cases} \alpha_{\mu_k}\psi_{n-1}^{\mu_1\cdots\mu_{k-1}} + \zeta_{\mu_k}\psi_{n-3}^{\mu_1\cdots\mu_{k-1}} & \\ \quad + k^{-1}\mathcal{B}_2\psi_{n-2}^{\mu_1\cdots\mu_k} & \\ \quad + 2k^{-1}\gamma_{\mu_{k+1}}\big\langle\psi_{n-1}^{\mu_1\cdots\mu_{k+1}\cdots\mu_k}\big\rangle & \text{for } k=1,\ldots,n-2 \\ \alpha_{\mu_k}\psi_{n-1}^{\mu_1\cdots\mu_{k-1}} & \text{for } k=n-1,n \end{cases} \tag{A.5}$$

and $\langle\cdots\rangle$ implies a symmetrization on the repeated index (i.e. the sum of the possible permutations of the repeated index, divided by the number of such terms). Note that, by definition, the quantity $a^+_{\mu_k}\cdots a^+_{\mu_1}|0\rangle$ equals the multidimensional Hermite polynomial of degree k, $H_k^{\vec{\mu}}(\boldsymbol{y})$.

At each step one must also impose the corresponding solvability condition, which forbids the presence of states proportional to $|0\rangle$ on every right-hand side of expressions (A.3), in order to avoid inversion problems. These constraints give

$$2\gamma_\mu\psi^\mu_{n+1}+\mathcal{B}_2\psi^\emptyset_n=0 \qquad \forall\, n\in\mathbb{N}. \tag{A.6}$$

By expressing each ψ^μ_{n+1}, through (A.5), as a function of a combination of $\psi^\emptyset_m$ for some $m\leqslant n$, expressions (A.6) are to be interpreted as equations for the quantities $\psi^\emptyset_n$. Together with the normalization condition $\int \mathrm{d}\boldsymbol{x}\,\psi^\emptyset_n\propto\delta_{n0}$, they can be solved recursively (numerically, or even analytically in some fortunate circumstances) once the basic flow $\boldsymbol{u}(\boldsymbol{x},t)$ is given. All such equations are of the advection–diffusion type and are forced by lower-order, already-solved quantities, with the exception of the equations for $n=0$ and $n=1$, which are homogeneous and thus imply $\psi^\emptyset_0=$ const. and $\psi^\emptyset_1=0$ under our assumptions. Moreover, one should note that even-order equations are forced only by even-order quantities, and the same happens with odd n's: this means that $\psi^\emptyset_{2n+1}=0\;\forall\, n\in\mathbb{N}$; therefore, our expansion (A.2) reduces to a series in integer powers of St only. In (10) we reported the first few equations corresponding to even n's, separating the components symmetric and antisymmetric in the vertical direction. Note that the spatial gradient and the flow field are vertically antisymmetric, while the vertical unit vector $\boldsymbol{G}$ (reminiscent of gravity) is symmetric in this sense. Hereafter we report the equation for $\psi^{\emptyset(\mathrm{a})}_6$, needed to compute the following term in Δw and which also requires knowledge of $\psi^{\emptyset(\mathrm{s})}_4$:

$$\begin{aligned}
\mathcal{M}\psi^{\emptyset(\mathrm{a})}_6 &= -(1-\beta)\mathrm{Fr}^{-2}G_\mu\partial_\mu\big(\psi^{\emptyset(\mathrm{s})}_4-\mathcal{N}_\nu\partial_\nu\psi^\emptyset_2\big)+\partial_\mu\mathcal{M}\mathcal{N}_\mu\psi^{\emptyset(\mathrm{a})}_4\\
&\quad-(1-\beta)^2\mathrm{Fr}^{-2}G_\mu\psi^\emptyset_0\partial_\nu(2\partial_\mu\mathcal{M}+\mathcal{M}\partial_\mu+\partial_\mu\mathcal{N}_\lambda\partial_\lambda/2)u_\nu\\
&\quad+2(1-\beta)^3\mathrm{Fr}^{-2}G_\mu\psi^\emptyset_0(\partial_\lambda u_\nu)(\partial_\mu\partial_\nu u_\lambda)\\
\mathcal{M}\psi^{\emptyset(\mathrm{s})}_4 &= \partial_\mu\mathcal{M}\mathcal{N}_\mu\psi^\emptyset_2-(1-\beta)\psi^\emptyset_0\partial_\mu\{[\partial_\nu\mathcal{M}\mathcal{N}_\nu/2-\mathcal{B}_2\mathcal{M}+\mathcal{N}_\nu\partial_\nu(\mathcal{N}_\lambda\partial_\lambda/2-\mathcal{B}_2)]\,u_\mu\\
&\quad+(\mathcal{N}_\lambda\partial_\lambda/2-\mathcal{B}_2)\partial_\nu\mathcal{N}_\mu u_\nu\},
\end{aligned} \tag{A.7}$$

where $\mathcal{N}_\mu\equiv\sqrt{2/\mathrm{Pe}}\alpha_\mu$, and $\mathcal{M}$ has already been defined in (11) (it equals $\mathcal{N}_\mu\partial_\mu-\mathcal{B}_2$).

Starting from the quantities $\psi^{\emptyset(\mathrm{a})}_n$ with even n, we can reconstruct the variation of the adimensionalized terminal velocity by substituting back (A.4) into (A.2) into (A.1) into (6), and by making use of the orthonormalization of the Hermite polynomials $H_k^{\vec{\mu}}(\boldsymbol{y})$ in the integral in the $\boldsymbol{y}$ variable with weight $p_\dagger^{1/2}(\boldsymbol{y})$ (note that $1=H_0^\emptyset$):

$$\begin{aligned}
\Delta\boldsymbol{w} &= \left(\frac{T}{L/U}\right)^{-1}\int\mathrm{d}t\int\mathrm{d}\boldsymbol{x}\int\mathrm{d}\boldsymbol{y}\,\boldsymbol{u}(\boldsymbol{x},t)p(\boldsymbol{x},\boldsymbol{y},t)\\
&=\frac{L}{UT}\int\mathrm{d}t\int\mathrm{d}\boldsymbol{x}\,\boldsymbol{u}(\boldsymbol{x},t)\int\mathrm{d}\boldsymbol{y}\,p_\dagger^{1/2}(\boldsymbol{y})\psi(\boldsymbol{x},\boldsymbol{y},t)\\
&=\frac{L}{UT}\int\mathrm{d}t\int\mathrm{d}\boldsymbol{x}\,\boldsymbol{u}(\boldsymbol{x},t)\int\mathrm{d}\boldsymbol{y}\,\mathrm{e}^{-y^2/2}\sum_{n=0}^{\infty}\mathrm{St}^{n/2}\psi_n(\boldsymbol{x},\boldsymbol{y},t)\\
&=\frac{L}{UT}\sum_{n=0}^{\infty}\mathrm{St}^{n/2}\int\mathrm{d}t\int\mathrm{d}\boldsymbol{x}\,\boldsymbol{u}(\boldsymbol{x},t)\int\mathrm{d}\boldsymbol{y}\,\mathrm{e}^{-y^2/2}\sum_{k=0}^{n}\frac{1}{k!}\psi_n^{\mu_1\cdots\mu_k}(\boldsymbol{x},t)a^+_{\mu_k}\cdots a^+_{\mu_1}|0\rangle\\
&=\frac{L}{UT}\sum_{n=0}^{\infty}\mathrm{St}^{n/2}\sum_{k=0}^{n}\frac{1}{k!}\int\mathrm{d}t\int\mathrm{d}\boldsymbol{x}\,\boldsymbol{u}(\boldsymbol{x},t)\psi_n^{\mu_1\cdots\mu_k}(\boldsymbol{x},t)\int\mathrm{d}\boldsymbol{y}\,\mathrm{e}^{-y^2/2}H_k^{\vec{\mu}}(\boldsymbol{y})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{L}{UT} \sum_{n=0}^{\infty} \mathrm{St}^{n/2} \sum_{k=0}^{n} \frac{1}{k!} \int \mathrm{d}t \int \mathrm{d}\boldsymbol{x}\, \boldsymbol{u}(\boldsymbol{x},t) \psi_n^{\mu_1\cdots\mu_k}(\boldsymbol{x},t)\delta_{k0} \\
&= \frac{L}{UT} \sum_{n=0}^{\infty} \mathrm{St}^{n/2} \int \mathrm{d}t \int \mathrm{d}\boldsymbol{x}\, \boldsymbol{u}(\boldsymbol{x},t) \psi_n^{\emptyset}(\boldsymbol{x},t) \\
&= \frac{L}{UT} \sum_{m=0}^{\infty} \mathrm{St}^{m} \int \mathrm{d}t \int \mathrm{d}\boldsymbol{x}\, \boldsymbol{u}(\boldsymbol{x},t) \psi_{2m}^{\emptyset}(\boldsymbol{x},t) \\
&= \frac{L}{UT} \sum_{m=0}^{\infty} \mathrm{St}^{m} \int \mathrm{d}t \int \mathrm{d}\boldsymbol{x}\, \boldsymbol{u}(\boldsymbol{x},t) \psi_{2m}^{\emptyset(\mathrm{a})}(\boldsymbol{x},t) \\
&= \frac{L}{UT} \sum_{m=2}^{\infty} \mathrm{St}^{m} \int \mathrm{d}t \int \mathrm{d}\boldsymbol{x}\, \boldsymbol{u}(\boldsymbol{x},t) \psi_{2m}^{\emptyset(\mathrm{a})}(\boldsymbol{x},t),
\end{aligned}
\tag{A.8}
$$

which corresponds to (9).

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